Four original problems

statements

The problems are listed in increasing order of difficulty.

Problem 1

Determine all functions $f : \mathbb{Q} \to \mathbb{Q}$ with the property that

$$f\big(xf(y) + 1\big) = y\,f(x) + 1$$

for all $x, y \in \mathbb{Q}$.

Problem 2

Let a, b be positive integers and let the sequence $(x_n)_{n\geq 0}$ be defined by $x_0 = a$, $x_1 = b$ and

$$x_{n+1} = \operatorname{lcm}(x_0, x_1, \ldots, x_n) + \gcd(x_0, x_1, \ldots, x_n), \quad n \geq 1.$$

Let p be a prime dividing some term x_n with $n \geq 3$. Prove that:

(a) if $p \mid \gcd(a, b)$, then p divides every term of the sequence;

(b) if $p \nmid \gcd(a, b)$, then p divides exactly one term of the sequence and, moreover, $p = 3$ or $p \equiv 1 \pmod 6$.

Problem 3

Let ABC be an acute triangle, let O be its circumcentre and let τ be the tangent at A to the circumcircle. Let B_1 and C_1 be the reflections of B and C respectively in the line τ, and let P, Q be the reflections of A in B and C respectively. Let M be the circumcentre of triangle OB_1C_1, and let X be the point on the ray (MA for which $MA \cdot MX = MO^2$. Prove that the points P, Q, O, X are concyclic.

Problem 4

Alex and Robin play the following game on a rectangular $m \times n$ board, with $m, n \geq 2$, using $k \geq 2$ colours numbered $0, 1, \ldots, k-1$. Initially every cell is uncoloured. Starting with Alex, the two players alternately strike one cell with a hammer: an uncoloured cell takes colour 0, and a cell of colour j becomes colour $j + 1 \pmod k$. Alex wins if, after some move, the board contains three cells of the same colour, two of which lie in the same row and two of which lie in the same column; Robin only tries to prevent this, for however long the game lasts.

Determine all triples (m, n, k) for which Alex has a winning strategy.