The Romanian Higher Algebra and Analysis Olympiad

Volume 1 : Analysis, 11th Grade

Sequences, limits, continuity, differentiability

Complete theory and problems

Vlad-Ștefan Constantinescu

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Volume 1 : Analysis, 11th Grade

Author: Vlad-Ștefan Constantinescu
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Foreword

About this volume. This is Volume 1 of a series of four volumes. It covers the mathematical analysis of the 11th grade (sequences, limits, continuity, differentiability), together with the bridge to the 12th grade. The volumes that accompany it are devoted to 12th grade analysis, linear algebra and abstract algebra.

The volume has two large components: **the theory**, with complete proofs and examples, and **the problems**, grouped into three levels of difficulty (Level 1 · Level 2 · Level 3). In the theory part I have included two sections which I consider essential and which are often missing from olympiad books: one on **the write-up** (how a rigorous proof is written and which errors of form the committee penalizes) and one on **constructions** (which objects to try when the statement gives you no direction).

The three levels. Throughout the book, the problems are classified into three levels, modelled on the three stages of the Romanian National Mathematical Olympiad: **Level 1** (the local stage, school and town, known in Romania as OLM), **Level 2** (the county stage, OJM) and **Level 3** (the national final, ONM). Problems are numbered accordingly: L1.5 is the fifth problem of Level 1. Broadly speaking, Level 1 problems are accessible competition problems, Level 2 corresponds to a solid regional olympiad, and Level 3 problems are at the level of a national final, comparable, for the subject matter of this book, with the hardest problems from student competitions such as the Putnam.

Analysis vs. algebra. Mathematical analysis is a continuous subject: the 12th grade material is nothing but a natural extension of the 11th grade one, and limits, integrals and series live in the same world. That is why I have built an explicit bridge between the two grades. Algebra stands differently: the algebraic structures of the 12th grade (groups, rings, fields, polynomials) form a coherent block in itself, with clear threads between them, but they have no connection with the linear algebra of the 11th grade. The two halves of algebra are separate worlds, and I have chosen to treat them as

such.

About the methods. I have deliberately mixed two perspectives. On the one hand, the techniques one meets at the olympiads in Romania, from the local stage up to the national one. On the other hand, some methods I met in the first two years of higher studies in the French CPGE system (the preparatory classes for the grandes écoles), generating functions, series-integral comparisons, asymptotics through singularities, and which I found perfectly usable at olympiad level, often clearer and more powerful than ad hoc approaches. The bridge between the 11th and the 12th grade is the place where the two perspectives meet most naturally.

About the syllabus. The levels $1 \cdot 2 \cdot 3$ mark *difficulty*, not strictly the syllabus of the corresponding stage. I have deliberately included, at a given level, problems using notions that do not figure officially in the syllabus of that stage: for instance problems with derivatives at Levels 1 and 2, even though, not being in the syllabus of the local and county stages, they cannot actually appear there. I have kept them as an exercise of the kind "and what if it did come up at a local or county stage": the basic idea is accessible at that level too, and training on them cannot hurt.

About the write-up. I have tried to write up the way it is required at the olympiad: I state and prove every lemma I use, I split the solutions into explicitly named steps, I check the hypotheses before applying a theorem and I mark clearly where each argument ends. The write-up is not a whim: in a competition, it makes the difference between a good idea and a full score.

About the problems. Almost all of them are invented by me. The exceptions (the classical problems and those taken from competitions) are indicated explicitly. I have also tried to include at least one problem for every result stated in the theory part: a lemma that does not "work" anywhere is not worth remembering.

Vlad-Ștefan Constantinescu
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I thank **Mr Flavian Georgescu** and **Mr Traian Preda**, who trained me for the olympiad and opened my access to techniques and perspectives that textbooks never touch.

I thank **Mr Cătălin Gherghe**, president of the Romanian Mathematical Society, organizer of the National Mathematical Olympiad and of the Senior Olympic Team. I thank him not only for the organizational effort he puts in, but also for the training he gives us at the Excellence Circle.

Part I

MATHEMATICAL ANALYSIS

Write-up and construction problems

Writing up analysis: from the picture to the proof

How to write up a solution, in general

Before anything specific to analysis, the rules that earn points on *any* paper:

1. **Announce the plan and the structure.** The first sentence says what you are going to prove and, ideally, how ("We show that f is constant, in two steps: first that it is periodic with arbitrarily small periods, then the conclusion by continuity."). Long steps become *named lemmas*, the grader marks against your scheme, not his.
2. **Every assertion has backing:** a displayed computation or a *named theorem, with its hypotheses checked explicitly* ("f is continuous on $[0,1]$ as a sum of continuous functions, hence by Weierstrass it attains its maximum"). "One sees that" and "obviously" are lost points, if it is obvious, the proof is short; write it.
3. **Objects are introduced before they are used**, with the correct quantifier: "Let $\varepsilon > 0$ be arbitrary", "*there exists* N such that...", "*choose*

$\delta = \min(\delta_1, \delta_2)$". The order of the quantifiers is mathematical content, not style, swapping them is the difference between continuity and uniform continuity.

4. **Notation: once defined, never recycled.** Do not use n both for the index of the sequence and for the degree of the polynomial; do not redefine the c from the mean value theorem as another c three lines below.
5. **Cases are exhausted in plain sight.** List them ("Case 1: $f'(x_0) > 0$..."), and write "case 2 is symmetric, replacing max by min" *only* if the symmetry is real and checkable in 10 seconds.
6. **On "determine all...": two mandatory halves.** *Verification* (the objects you found really do satisfy the condition, explicit computation) and *classification* (there are no others). The mark scheme almost always separates them; handing in only the verified guess means handing in half a problem.
7. **The scratch work does not go into the paper.** The route by which you found the solution (pictures, small cases, heuristics, the compass equation from asymptotics) stays on the scratch paper; what goes onto the final copy is only the logical chain, in logical order, not in the order of discovery. One complete page beats three pages of searching.
8. **The last 5 minutes: the audit.** Endpoints of intervals and closed/open-type hypotheses, degenerate cases ($n = 1$, the constant function, $a = b$), constants of integration, signs when squaring, exactly the places in the lists of traps below.

The graphical method: from the picture to the proof

Many analysis problems are *solved* like this: you draw a graph, you read the condition in the statement off the picture, you "see" the solution in the drawing, and then you go back to the problem. This is the right method for **finding** the solution, the graph is the fastest generator of intuition in analysis. But this is also the trap: **a solution written up on a picture usually gets at most 2 points out of 7.** The graph is scaffolding, not a proof: the committee does not give credit for "one sees from the figure", but for the chain of theorems that replaces the figure.

So the correct workflow has *two passes*:

1. **The discovery pass (on scratch paper):** you draw, you interpret the condition geometrically (an inequality = a relative position of graphs; an

equation = an intersection; a mean = an area or a slope), you guess the configuration and the solution.

2. **The write-up pass (on the final copy):** you translate *every* visual assertion into the statement of a theorem. The standard dictionary: "the graph goes up" → monotonicity proved from the sign of the derivative (on an *interval*!); "the graphs cross" → Cauchy-Bolzano (the intermediate value theorem) applied to a difference function, with the signs at the endpoints computed explicitly; "the graph stays above the tangent/chord" → convexity, with the tangent line inequality or the slope lemma cited; "it approaches..." → a limit with ε or a squeeze; "the average area/slope" → a mean value theorem or Lagrange, with the hypotheses checked. If a sentence in your write-up has no theorem from the notes or no computation behind it, it does not exist for the grader.

Example

Determine all differentiable functions $f : \mathbb{R} \to \mathbb{R}$, with continuous derivative, such that

$$f(x) = \max\big(f'(x),\ x+1\big), \qquad \forall x \in \mathbb{R}.$$

Pass 1 (discovery, on the picture).

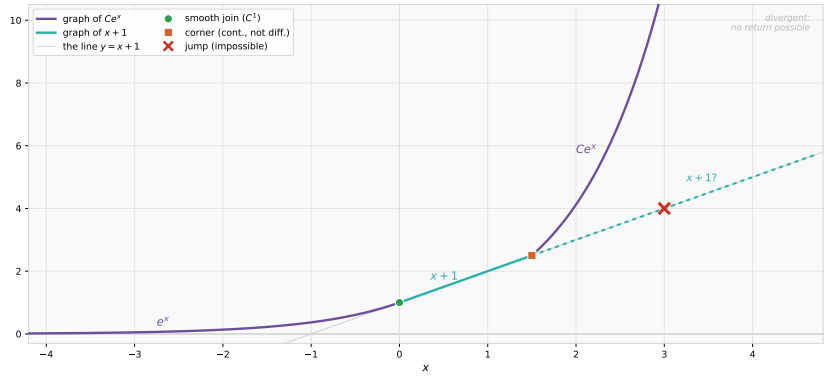
Key observation. From the continuity of f': if $f'(x_0) > x_0 + 1$ strictly, then $f'(x) > x + 1$ on a whole interval around x_0 (and conversely). So on each maximal interval we are in exactly one of the situations:

- $f'(x) > x + 1$: then $f(x) = f'(x)$, hence $f' = f$, that is $f(x) = Ce^x$ on that interval;
- $f'(x) \le x + 1$: then $f(x) = x + 1$, hence $f'(x) = 1$, which implies $f'(x) < x + 1$ (correct if $x \neq 0$), hence $f(x) = x + 1$.

In other words, f is **piecewise**: it alternates between exponential stretches Ce^x and linear stretches $x + 1$.

What could happen? The graph below shows a function alternating linear → exponential → linear → exponential → linear. The question: can these pieces be glued together consistently?

Why does the alternation fail? At a transition from exponential → linear we need $Ce^{x_1} = x_1 + 1$ and $Ce^{x_1} = 1$ (a C^1 gluing), that is, precisely tangency $e^{x_1} \ge x_1 + 1$, with equality. The only solution: $C = 1$, $x_1 = 0$. But for $x > 0$, $e^x > x + 1$ strictly, so the exponential never comes back down to the line. Any transition after $x = 0$ is impossible. What remains



A candidate piecewise function: it alternates between $y = x + 1$ (blue) and Ce^x (purple). The arrows show the transition points. The key: once the exponential overtakes the line, it can never come back to it, since $Ce^x > x + 1$ for x large enough.

is only: the family Ce^x (with $C \geq 1$, so that $\geq x+1$ always holds) and the solution glued exactly once at $x = 0$.

Pass 2 (the write-up). Let $U = \{x : \ f'(x) < x + 1\}$, an *open* set, since f' is **continuous** (this is where the hypothesis is used!).

On U: $f(x) = \max(f'(x), x+1) = x+1$, hence, U being open, $f'(x) = 1$ on U. But then the membership condition $f'(x) < x+1$ becomes $1 < x+1$, that is $x > 0$: therefore $U \subseteq (0, \infty)$.

Case $U = \varnothing$: then $f'(x) \geq x + 1$ everywhere, hence $f(x) = f'(x)$ for every x. The equation $f' = f$ gives $f(x) = Ce^x$ (via the auxiliary function $f(x)e^{-x}$, as in the section on differential equations). There remains the constraint $Ce^x \geq x + 1$ for every x: for $C \leq 0$ it fails at $x = 0$; for $C > 0$, the minimum of the function $Ce^x - x - 1$ is attained at $x = -\ln C$ and equals $\ln C$, so the constraint is equivalent to $C \geq 1$. We obtain the family $f(x) = Ce^x$, $C \geq 1$.

Case $U \neq \varnothing$: let (a, b) be a component of U, with $0 \leq a < b \leq \infty$. On (a, b), $f' \equiv 1$; by the continuity of f', $f'(a) = 1$. But $a \notin U$, hence $f'(a) \geq a + 1$, that is $1 \geq a + 1$, so $a \leq 0$: combined with $a \geq 0$, this gives $a = 0$. If $b < \infty$: likewise $f'(b) = 1 \geq b+1$, hence $b \leq 0$, contradiction with $b > 0$. Therefore $U = (0, \infty)$, and $f(x) = x + 1$ on $[0, \infty)$ (with $f(0) = 1$ by continuity). On $(-\infty, 0]$, the complement of U, we have $f = f'$ (as above), hence $f(x) = Ce^x$ there; $f(0) = 1$ gives $C = 1$. Final checks: the gluing is of class C^1 ($f'(0-) = e^0 = 1 = f'(0+)$), and on $(-\infty, 0)$ the constraint $f'(x) = e^x \geq x + 1$ is classical. ■

Answer: $f(x) = Ce^x$ with $C \geq 1$, *or* $f(x) = \begin{cases} e^x, & x \leq 0 \\ x + 1, & x \geq 0. \end{cases}$

What the write-up bought: the picture *guessed* both families, but only the argument with the open set U proves that **there are no others**, exactly the part that is invisible on the graph. Note also the dictionary at work: "the tangency of e^x to the line" $\to$ the minimum of $Ce^x - x - 1$ computed with the derivative; "the gluing works" $\to$ the explicit check C^1; "from 0 on, only the line" $\to$ the argument with the endpoints of the components of U.

Frequent traps (11th grade)

1. Passing to the limit in a recurrence requires *first* proving convergence; $\ell = f(\ell)$ alone proves nothing.
2. Lagrange/Rolle require continuity on the *closed* interval and differentiability on the *open* one, check the hypotheses explicitly.
3. The consequences of Lagrange (monotonicity from the sign of the derivative) hold only on *intervals*, not on unions of intervals (e.g. $-1/x$ has positive derivative on $\mathbb{R}^*$ but is not increasing on $\mathbb{R}^*$).
4. L'Hôpital: check the indeterminate form and the existence of the limit of the ratio of the derivatives; the implication does not reverse.
5. The intermediate value property does not imply continuity; continuity implies it.
6. $a_{n+1} - a_n \to 0$ does not imply convergence; $\frac{a_{n+1}}{a_n} \to 1$ decides nothing.
7. Extremum points can also occur at the endpoints of the domain or at points of non-differentiability, Fermat covers only interior points where the derivative exists.
8. For limits with $\limsup$ / $\liminf$ or with subsequences: convergence along (x_{2n}) and along (x_{2n+1}) must give the *same* limit.

Traps with integrals (12th grade)

1. "Continuous $\iff$ has antiderivatives" is false in both refinements: continuity is sufficient but not necessary; the intermediate value property is necessary but not sufficient.
2. Leibniz-Newton requires G to be an antiderivative on the *whole* interval, beware of "piecewise" antiderivatives with mismatched constants at the gluing (a classic with $\int \frac{dx}{2+\cos x}$ under the substitution $t = \operatorname{tg} \frac{x}{2}$, discontinuous at π).
3. In a change of variable for definite integrals the endpoints change too; u must be differentiable with continuous derivative on the interval of integration.
4. $\int_a^b f = 0$ does not imply $f \equiv 0$ without the hypotheses $f \geq 0$ *and* f continuous.
5. Riemann integrability requires boundedness; $\int_0^1 \frac{dx}{\sqrt{x}}$ is an *improper* integral, not an ordinary Riemann one, it is handled by a limit.
6. In Riemann sums, the function must be integrable on the closed interval; singularities at the endpoints (as with $x^{-\alpha}$) require an additional truncation argument.

Constructions: which functions to try

Note: this section comes at the beginning because it is an *attitude*, not a chapter, it is used everywhere. Its examples invoke notions treated rigorously later on (derivatives, differential equations, gluings on $\mathbb{Q}$, improper integrals); on a first reading you may skip the proofs and keep the reflexes.

Construction problems at the ONM are often of the type *"prove that there exists a function with the properties..."* or *"give an example of a function which..."*. The main idea: to recognize quickly **what shape of function stands a good chance of working**, and then to determine the parameters by direct computation.

General principles of construction

Construction problems look the same in every branch, combinatorics, number theory, analysis, and share a common folklore of strategies (adapted here from Cody Johnson's "Olympiad Constructions" and the olympiad combinatorics literature):

1. **Small cases, always first.** Solve the problem for $n = 1, 2, 3, 4$: the small cases guess the answer for you (the minimum/maximum, the shape) and, more importantly, they show you the *pattern* along which the general construction is assembled. The analysis analogue: try the toy examples from the catalogue first.
2. **Construction by induction/assembly.** From a construction of size n one often manufactures one of size $n + 1$ or $2n$ (you double it, you glue, you add a piece). In analysis: gluing functions piecewise (e^x with $x + 1$ at ONM 1994), sequences of functions defined recursively.
3. **Prove first, then search: narrowing by constraints.** Before searching blindly, prove *mandatory properties* of any solution (in analysis: signs, monotonicity, values forced at certain points). Every constraint cuts down the search space, and often the constraints, taken together, *dictate* the construction. This is exactly the "structure guesses" of our sections: at $(f')^2 + 1 = f$, comparing degrees decided everything before any attempt.
4. **On "prove or disprove": attack the NO branch seriously.** Genuinely try to prove impossibility; *the precise place where your argument breaks down* is the map towards the construction, that is where the freedom lies that a construction has to exploit. And the contest discipline: if in the first half of the time you have not proved "yes", spend the second half on "no", without oscillating.
5. **The simplicity principle.** If a construction exists, there is almost surely one made of *simple pieces*: elementary moves, repeated blocks, functions from the catalogue. Do not look for exotic objects before exhausting the trivial combinations, the moral of all the examples in these notes, where the answers were e^x, $|x|$, Dirichlet, $\sqrt{x^2 + \varepsilon}$, never anything unheard of.

The rest of the section applies these principles to the specifics of analysis: which *families of functions* play the role of the "simple pieces".

Reflex 1: for functional or differential equations, try a polynomial first

The first reflex is to assume that the function is a polynomial and to see *what degree it would have to have*: comparing degrees tells you very quickly whether there is any chance at all. The rules for degrees: differentiation lowers the degree by 1; squaring doubles it; composition multiplies them.

Example. Construct f with $\big(f'(x)\big)^2 + 1 = f(x)$ for every x.

Step 1 (the degree). If f is a polynomial of degree n: f' has degree $n-1$, hence $(f')^2$ has degree $2(n-1)$. Equality of degrees requires $2(n-1) = n$, that is $n = 2$. We try $f(x) = ax^2 + bx + c$.

Step 2 (direct computation). $f'(x) = 2ax + b$, hence

$$(2ax+b)^2+1 = ax^2+bx+c \quad \Longleftrightarrow \quad 4a^2x^2+4abx+b^2+1 = ax^2+bx+c,$$

and identifying coefficients gives the system $4a^2 = a$, $4ab = b$, $b^2+1 = c$.

Step 3 (solving). From the first: $a = 0$ (which loses degree 2) or $a = \frac{1}{4}$. For $a = \frac{1}{4}$, the second equation becomes $b \cdot 0 = 0$, so b is *arbitrary*, and $c = b^2 + 1$:

$$\boxed{f(x) = \frac{x^2}{4} + bx + b^2 + 1, \qquad b \in \mathbb{R}.}$$

Check: $f'(x) = \frac{x}{2} + b$, $\left(\frac{x}{2} + b\right)^2 + 1 = \frac{x^2}{4} + bx + b^2 + 1 = f(x)$. ✓

Reflex 2: the condition $f > 0$ on the whole domain

When the statement demands $f(x) > 0$ for every x in the domain (usually $(0, \infty)$), the natural shapes to test are

$$f(x) = c_2\, x^{c_1} \qquad \text{and} \qquad f(x) = b \cdot a^x,$$

with $c_2, b > 0$, powers and exponentials are the "canonical" positive functions and they behave perfectly under multiplication, composition and differentiation (they stay in the same family). This is no accident: these are exactly the solutions of the classical functional equations (the section on Cauchy's equations), so any condition with multiplicative structure will select them.

Reflex 3: controlled discontinuities, Dirichlet-type functions

When you have to construct a function with a *controlled* number of points of discontinuity or of non-differentiability, the standard idea is to define it differently on $\mathbb{Q}$ and on $\mathbb{R} \setminus \mathbb{Q}$. The theoretical tool is already in the notes (the subsection "Functions defined by cases on $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$"): for g, h continuous, the glued function is **continuous exactly at the points where** $g(x) = h(x)$, and differentiable exactly where the branches meet together with their derivatives. The construction thus becomes an *equation*: you want discontinuity exactly on a set S? Choose branches that agree exactly on the complement of S.

Reflex 4: which classical functions satisfying the hypothesis do I know?

Before inventing something complicated, ask yourself: *"which example with these properties do I already know?"* Olympiad subquestions of the type "give an example / does it follow that...?" are almost always solved from a short catalogue, combined with small adjustments (translations, multiplication by constants, $x^2 + \varepsilon$ instead of x^2):

1. **Dirichlet** $\mathbf{1}_{\mathbb{Q}}$, discontinuous everywhere; $x \cdot \mathbf{1}_{\mathbb{Q}}$, continuous only at 0; x on $\mathbb{Q}$, $x + 1$ otherwise, discontinuous everywhere but with limits "along the branches".
2. $|x|$, continuous, not differentiable at one point; $\sqrt{x^2 + \varepsilon}$, the smooth function that approximates it.
3. $\sin \frac{1}{x}$, bounded without a limit at 0; $x \sin \frac{1}{x}$, continuous, not differentiable at 0; $x^2 \sin \frac{1}{x}$, differentiable with discontinuous derivative.
4. x^3, strictly increasing with derivative vanishing at a point; $x + 2 \sin x$, a derivative changing sign infinitely many times, while the function $\to \infty$.

The moral: for part b) questions of the type "does it follow that...?", the answer "no" is proved with a member of the catalogue, minimally adjusted.

Example

Let $f:\mathbb{R}\to\mathbb{R}$ be such that there exist a differentiable function $g:\mathbb{R}\to\mathbb{R}$ and a sequence (a_n) of strictly positive numbers, convergent to 0, such that

$$g'(x)=\lim_{n\to\infty}\frac{f(x+a_n)-f(x)}{a_n}\qquad\forall x\in\mathbb{R}.$$

a) Give an example of such an f which is nowhere differentiable.

Solution. Let us think simply: we need a nowhere differentiable function, the first candidate from the catalogue is Dirichlet, $f=\mathbf{1}_{\mathbb{Q}}$. Let us check that it works with $a_n=\frac{1}{n}$: if $x\in\mathbb{Q}$, then $x+\frac{1}{n}\in\mathbb{Q}$; if $x\notin\mathbb{Q}$, then $x+\frac{1}{n}\notin\mathbb{Q}$, adding a rational number *preserves the class*. Hence

$$f\Big(x+\frac{1}{n}\Big)=f(x)\quad\forall x,n\qquad\Longrightarrow\qquad\frac{f(x+a_n)-f(x)}{a_n}=0\to 0,$$

so the hypothesis is satisfied with g constant ($g'\equiv 0$). And f is discontinuous at every point (the constant branches 1 and 0 never meet, the gluing corollary), hence nowhere differentiable. ■

Worth remembering: the whole solution lies in the choice of the sequence a_n *rational*, the limit along one particular sequence does not see the fine structure of the function. It is exactly this phenomenon that makes part b) of the problem (if f is *continuous*, then it is differentiable) the hard part.

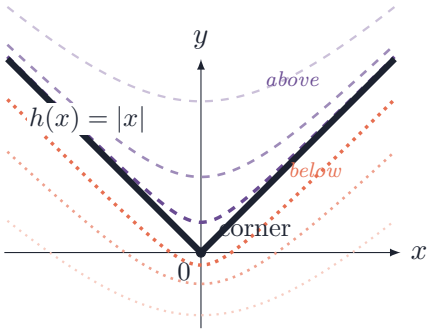
Example: ONM 2026, national round, 11th grade, Problem 1b)

Statement. Let $h:\mathbb{R}\to\mathbb{R}$ be a continuous function and let (f_n), (g_n) be two sequences of differentiable functions with the property that $f_n(x)<h(x)<g_n(x)$ for every $x\in\mathbb{R}$ and every $n\in\mathbb{N}^*$, and that $g_n(x)-f_n(x)\to 0$ for every $x\in\mathbb{R}$. Is h necessarily differentiable?

Answer: NO.

I took part in ONM 2026 and this was my reasoning on this problem.

How to guess the counterexample. During the contest I drew a graph. The conditions of the problem say that h is caught between two families of differentiable functions that squeeze it, on the graph this means two "pincers" of smooth curves coming down and up towards h from both sides. The only continuous nowhere differentiable function I knew was $h(x)=|x|$, it has a corner at 0, but is otherwise perfectly harmless. I drew it and looked for differentiable curves, above and below, that would squeeze it:



The shape of the curves above immediately suggested a smoothed hyperbola to me. Once I had chosen h and noticed that the curves resemble $\sqrt{x^2+\varepsilon}$, I chose formally:

The construction. Put

$$g_n(x)=\sqrt{x^2+\tfrac{1}{n}}, \qquad f_n(x)=\sqrt{x^2+\tfrac{1}{n}}-\sqrt{\tfrac{2}{n}}.$$

Differentiability: $x^2+\frac{1}{n}>0$, hence both are C^∞ on $\mathbb{R}$.

The squeeze $f_n<h<g_n$: The inequality $g_n>|x|$ is clear. The inequality $f_n<|x|$ amounts to $\sqrt{x^2+\frac{1}{n}}-\sqrt{\frac{2}{n}}<|x|$, that is (squaring, both sides being nonnegative for n large):

$$x^2+\frac{1}{n}<x^2+2|x|\sqrt{\tfrac{2}{n}}+\frac{2}{n} \quad\Longleftrightarrow\quad 0<2|x|\sqrt{\tfrac{2}{n}}+\frac{1}{n},$$

which is true for every $x\in\mathbb{R}$.

The squeezing: $g_n(x)-f_n(x)=\sqrt{\frac{2}{n}}\to 0$ uniformly.

All the hypotheses are satisfied, but $h=|x|$ is not differentiable at 0. ■

Part II

SIX PROBLEMS FROM THE BOOK

Level 1 is the local round, Level 2 the county round, Level 3 the national round. Each statement is followed by its full solution, exactly as in the book.

Problem L1.18

Determine all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the property that

$$f(x+1) = f(x) + 1 \qquad \text{and} \qquad f\big((x+1)^2\big) = f(x^2) + 2x + 1$$

for every $x \in \mathbb{R}$.

Solution. We write $f(x) = x + g(x)$ and reformulate the conditions:

(1) $g(x+1) = g(x)$ for every x (from the first condition), hence g is *continuous and periodic* with period 1.

(2) $g\big((x+1)^2\big) = g(x^2)$ for every x (from the second condition, after simplification).

We must show that g is constant. Replacing x by $\sqrt{t} - 1$ (for $t > 0$) in (2):

$$g(t) = g\Big((\sqrt{t} - 1)^2\Big), \qquad t > 0. \tag{$*$}$$

Step 1. For $t > 1$: $\sqrt{t} > 1$, hence $(\sqrt{t} - 1)^2 < t$ and the sequence $t_0 = t$, $t_{n+1} = (\sqrt{t_n} - 1)^2$ is strictly decreasing, so $g(t) = g(t_1) = g(t_2) = \cdots$ Since the sequence converges to a fixed point of $\varphi(t) = (\sqrt{t} - 1)^2$, which has the fixed points 0 and $\frac{1}{4}$ (the solutions of $\varphi(t) = t$, that is $\sqrt{t} - 1 = \pm\sqrt{t}$), and 0 is the stable fixed point, $t_n \to 0$. By continuity $g(t) = g(0)$.

Step 2. By periodicity, $g(t) = g(0)$ for every $t > 0$ implies $g = g(0)$ on all of $\mathbb{R}$.

Therefore $g \equiv c$ (constant), and

$$\boxed{f(x) = x + c, \qquad c \in \mathbb{R}.}$$

Check. $f(x+1) = x + 1 + c = (x + c) + 1 = f(x) + 1$ ✓; $f((x+1)^2) = (x+1)^2 + c = x^2 + 2x + 1 + c = f(x^2) + 2x + 1$ ✓.

Problem L2.10

Let $f : [0, 1] \to \mathbb{R}$ be differentiable. Show that there exists $c \in (0, 1)$ such that

$$\pi\,(1 + c^2)\,f'(c) \;=\; 4\big(f(1) - f(0)\big).$$

Solution. We apply Cauchy's theorem, the generalized mean value theorem (Theorem 2.73), to the pair (f, g), where

$$g(x) = \arctan x.$$

The function g is differentiable on $[0,1]$, with $g'(x) = \dfrac{1}{1+x^2} \neq 0$ on $(0,1)$. The theorem gives a $c \in (0,1)$ with

$$\frac{f(1)-f(0)}{g(1)-g(0)} = \frac{f'(c)}{g'(c)} \Longrightarrow \frac{f(1)-f(0)}{\frac{\pi}{4}-0} = (1+c^2)\, f'(c),$$

which is exactly the required identity.

Remark 1. The choice of g is a "change of clock": Lagrange (2.71) is the particular case $g(x) = x$. With other clocks one obtains other identities, for instance $g(x) = x^2$ on $[a,b] \subset (0,\infty)$ gives $f(b) - f(a) = \dfrac{f'(c)}{2c}\,(b^2 - a^2)$.

Remark 2. Nothing can be said about the *position* of c: for $f = g = \arctan$ every c works, whereas for $f(x) = x$ the point is unique ($\pi(1+c^2) = 4$, hence $c = \sqrt{4/\pi - 1} \approx 0{,}52$).

Problem L2.15

Are there functions $f : \mathbb{R} \to \mathbb{R}$ that are periodic with arbitrarily small periods? If moreover f is continuous, is it still possible?

Solution. **Without continuity: YES.** Dirichlet's function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q}, \\ 0 & x \notin \mathbb{Q} \end{cases}$$

has every rational $r > 0$ as a period: $f(x+r) = f(x)$ for every x, since $x \in \mathbb{Q} \iff x + r \in \mathbb{Q}$. Since $\mathbb{Q}$ is dense, there are arbitrarily small periods $r > 0$.

With continuity: NO; f is constant. Let f be continuous with periods $p_n \to 0$, $p_n > 0$. Let $x, y \in \mathbb{R}$ and $\varepsilon > 0$. By continuity at x there exists $\delta > 0$ with $|f(z) - f(x)| < \varepsilon$ for $|z - x| < \delta$. We choose n with $p_n < \delta$. We take $k = \lfloor (y-x)/p_n \rceil$ (rounding to the nearest integer); then $|y - kp_n - x| < p_n < \delta$. By periodicity $f(y) = f(y - kp_n)$, hence $|f(y) - f(x)| < \varepsilon$. Since ε, y are arbitrary, f is constant.

Problem L2.16

(a) Are there continuous functions differentiable *only* at the rational points?

(b) Are there continuous functions differentiable *only* at the irrational points?

Solution. **(a) NO.** If f is continuous and differentiable at every rational, the sets

$$A_n = \{x \in \mathbb{R} : |f(x+h) - f(x)| \le n|h| \text{ for every } |h| \le 1/n\}$$

are *closed* and cover $\mathbb{Q}$ (differentiability implies the local boundedness of the difference quotient). By Baire's theorem ($\mathbb{R}$ is a complete Baire space), since $\mathbb{Q} \subseteq \bigcup_n A_n$ is dense, at least one A_{n_0} has nonempty interior: it contains an interval (c, d). On (c, d), the function f is Lipschitz with constant n_0. By Lebesgue's theorem (Lipschitz functions are differentiable almost everywhere), f is differentiable at almost every point of (c, d), hence in particular at irrational points. Contradiction.

(b) YES, and remarkably so. Let $(q_n)_{n\ge1}$ be an enumeration of the rationals. The function

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} \, |x - q_n|^{1/2}$$

is *continuous* (uniform convergence on every compact set). At any rational q_k: the term $n = k$ contributes $2^{-k}|x - q_k|^{1/2}$, which is not differentiable at q_k (the quotient $|h|^{1/2}/h = |h|^{-1/2} \operatorname{sgn} h \to \pm\infty$); the sum of the other terms *is* differentiable at q_k, hence f is not differentiable at q_k. At any irrational α: each term is differentiable at α, and the series of derivatives converges absolutely (the coefficients 2^{-n} decrease geometrically against the $|\alpha - q_n|^{-1/2}$, which are controllable), hence f is differentiable at α.

Remark Result (b) is surprising: the set of irrationals *can* be the exact differentiability set of a continuous function. Result (a) shows that $\mathbb{Q}$, although it is F_σ (a countable union of closed sets), cannot be the differentiability set of any continuous function.

Problem L3.8

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with the property

$$f(x) = 1 \quad \Longleftrightarrow \quad f'(x) = 0 \qquad (\text{for every } x \in \mathbb{R}).$$

Show that every value $v \neq 1$ is attained by f at at most two points.

Solution. Denote by

$$A = \{x \in \mathbb{R} : \ f(x) = 1\} = \{x \in \mathbb{R} : \ f'(x) = 0\}$$

the common set from the hypothesis (the two descriptions coincide, precisely by the given equivalence).

Step 1: A is a closed interval. Closedness is immediate: $A = f^{-1}\big(\{1\}\big)$, and f is continuous (being differentiable).

We show that A has no "holes". Let $x_1 < x_2$ be two points of A; we prove that $f \equiv 1$ on $[x_1, x_2]$, whence $[x_1, x_2] \subseteq A$.

The function f is continuous on the compact set $[x_1, x_2]$, hence it attains its bounds there (Weierstrass, Theorem 2.54): let M be the maximum and m the minimum.

Suppose $M > 1$. Since $f(x_1) = f(x_2) = 1 < M$, the maximum is *not* attained at the endpoints, hence it is attained at an *interior* point $d \in (x_1, x_2)$. Being an interior local extremum point of a differentiable function, Fermat's theorem (Theorem 2.68) gives $f'(d) = 0$; by hypothesis, then $f(d) = 1$. But $f(d) = M > 1$, a contradiction. Hence $M \leq 1$.

Symmetrically, if $m < 1$, the minimum is attained at an interior point e, where Fermat gives $f'(e) = 0$, hence $f(e) = 1$; but $f(e) = m < 1$, a contradiction. Hence $m \geq 1$.

From $m \geq 1$ and $M \leq 1$ it follows that $f \equiv 1$ on $[x_1, x_2]$. Therefore A is an interval (possibly empty, reduced to a point, a half-line or even $\mathbb{R}$).

Step 2: the complement has at most two components. Being a *closed* interval, the complement $\mathbb{R} \setminus A$ is a union of at most *two* open intervals, a "ray" to the left of A and one to the right:

$$\mathbb{R} \setminus A \;\subseteq\; (-\infty, \alpha) \;\cup\; (\beta, +\infty),$$

where $\alpha = \inf A$ and $\beta = \sup A$ (if $A = \emptyset$, the complement is $\mathbb{R}$ itself, that is, a single component; if $A = \mathbb{R}$, the complement is empty).

Step 3: on each component, f is strictly monotone. Let J be one of these components (an open interval). On J, the derivative f' *does not vanish*: a zero of f' would be, by hypothesis, a point at which $f = 1$, hence a point of A, and $J \cap A = \emptyset$.

The derivative has the intermediate value property (Theorem 2.74): if it took on J both negative and positive values, it would have to pass through 0, excluded. Hence f' keeps a *constant sign* on J, and by the consequence of Lagrange's theorem (Theorem 2.71), f is strictly monotone on J. In particular, f is *injective* on J: the equation $f(x) = v$ has at most one solution in J.

Step 4: the count. Let $v \neq 1$ and let x be a solution of the equation $f(x) = v$. Then $x \notin A$ (since on A the function equals $1 \neq v$), hence x belongs to one of the at most two components of $\mathbb{R} \setminus A$. By Step 3, each component contains at most one solution. Therefore the equation $f(x) = v$ has

cel mult 2 solutions.

Remark 1 (the bound is attained). For $f(x) = 1 - x^2$ we have $f(x) = 1 \iff$

$x = 0$ and $f'(x) = -2x = 0 \iff x = 0$: the hypothesis is satisfied. Every $v < 1$ is attained at exactly two points, $x = \pm\sqrt{1-v}$. So "two" cannot be lowered to "one": such a function is *not* necessarily injective.

Remark 2 (when is f injective). The proof yields the exact criterion: f is injective if and only if A has at most one point *and* f' keeps the *same* sign on either side of A. Compare:

$$f(x) = 1 + x^3 \quad \text{(injective)}, \qquad f(x) = 1 - x^2 \quad \text{(non-injective)},$$

both with $A = \{0\}$, the only difference is the sign of the derivative around the origin.

Remark 3 (a trap). It is tempting to believe that f cannot *cross* the level 1: every point with $f(x) = 1$ being a critical point, the graph has a horizontal tangent there. False, the crossing can happen through an *inflection*: $f(x) = 1 + x^3$ passes from values < 1 to values > 1, even though $f'(0) = 0$.

Remark 4 (the extrema). Again from Fermat: *every* local extremum of such a function has value 1. The level 1 is the only one at which the graph can "turn back", the hypothesis controls exclusively the structure of this level, and nothing else.

Problem L3.10

Let $a > 0$ and let $f : [0, a) \to \mathbb{R}$ be differentiable, with $f(0) = 1$ and $f'(x) = f(x)^2$ for every $x \in [0, a)$. Show that $a \leq 1$ and that $f(x) = \dfrac{1}{1-x}$.

Solution. *f does not vanish.* Suppose the contrary and let $c = \inf\{x \in [0, a) : f(x) = 0\}$; by continuity $f(c) = 0$, and $f > 0$ on $[0, c)$ (since $f(0) = 1 > 0$). On $[0, c)$, the function $g = \dfrac{1}{f}$ is differentiable, with

$$g' = -\frac{f'}{f^2} = -1;$$

then $(g(x) + x)' = 0$ and, by Corollary 2.72, $g(x) + x$ is constant:

$$g(x) = g(0) - x = 1 - x \Longrightarrow f(x) = \frac{1}{1-x} \quad (0 \leq x < c).$$

But then: if $c < 1$, continuity gives $f(c) = \lim_{x \to c^-} \frac{1}{1-x} = \frac{1}{1-c} \neq 0$, a contradiction; and $c \geq 1$ is impossible, since $f(x) = \frac{1}{1-x} \to +\infty$ for $x \to 1^-$, whereas f, being defined and continuous at $1 \leq c < a$, would have to have a finite limit there. Hence f does not vanish, it remains strictly positive, and the reasoning above (valid now on the whole domain on which $f > 0$, that is, on all of $[0, a) \cap [0, 1)$) gives $f(x) = \dfrac{1}{1-x}$.

$a \le 1$. If $a > 1$, the point $x = 1$ lies in the domain; f is continuous at 1, but $f(x) = \frac{1}{1-x} \to +\infty$ for $x \to 1^-$, impossible for a real-valued function. Hence $a \le 1$ and $f(x) = \dfrac{1}{1-x}$ on $[0, a)$.

Remark. The phenomenon is called *blow-up in finite time*: the quadratic growth in $f' = f^2$ is too fast for a global solution, unlike $f' = f$, whose solutions Ce^x live on all of $\mathbb{R}$.

End of the free sample

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